Non-Hermitian quantum canonical variables and the generalized ladder operators

W. S. l'Yi[1]

Department of Physics

Ch'ungbuk National University

Ch'ongju, 360-763, Ch'ungbuk, Korea

ABSTRACT

Quantum canonical transformations of the second kind and the non-Hermitian realizations of the basic canonical commutation relations are investigated with a special interest in the generalization of the conventional ladder operators. The opeator ordering problem is shown to be resolved when the non-Hermitian realizations for the canonical variables which can not be measured simultaneously with the energy are chosen for the canonical quantizations. Another merit of the non-Hermitian representations is that it naturally allows us to introduce the generalized ladder operators with which one can solve eigenvalue problems quite neatly.

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[1]wslyi@cbucc.chungbuk.ac.kr

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I. Introduction

Canonical transformations which include, as parts, both point transformations and time evolutions are not only theoretically but also practically important concepts for solving classical problems. Constructions of practically useful quantum versions of the canonical transformations are, on the other hand, as elusive as solving the quantum mechanical equations of motions themselves. Inspired by the beauty of the classical canonical transformations there appeared several interesing attempts. The widely known endeavor relied on the Hamiltonian path integral qunatization techniques [1]. The virtue of this formalism is that all the physical quantities are pure numbers and there are no operator ordering problems. But there are still pitfalls in this approach. One is that the canonical momenta p_i and p_{i+1} of a path integral at time slices t and $t + \epsilon$ are unrelated. In other word $q_i(p_{i+1} - p_i)$ is longer $\epsilon q_i \dot{p}_i$, and all the complications arise.

There are another attempts which may possibly circumvent this problem. The "effective generating technique" by some authors [2] is one of the proposals. In this approach the quantum generating function of a canonical transformation is written in terms of a series expansion in powers of $\hbar$ whose leading order term is the corresponding classical one. On the other hand Anderson extended unitary canonical transformations to non-unitary ones [3]. Even though it is quite general it lacked clear classical analogy. To improve this weak point another proposal [4] based on the more traditional "mixed matrix element technique" [5] is presented. In that paper it is shown that to get useful quantum canonical transformations one should allow non-Hermitian representations for the various canonical variables. It is clear that for the dynamical variables

which can not be measured with energy simultaneously one may freely choose non-Hermitian representations. In this non-Hermitian operator technique the classical analogy is preserved upon quatization, and higher $\hbar$ power terms of the effective generating function are interpreted as non-Hermitian mordifications to the Hamiltonian operator corresponding to the classically transformed Hamiltonian. But this operator technique of canonical transformation has both advantages and disadvantages. The facts that this surmounts the operator ordering problems by the concept of "well-orderedness" and that whenever there is a classically useful generating function there is a high probability to solve the Schrödinger equation, cause us to show favor to this approach. In that paper various kinds of the quantum canonical transformations inspired by the classical counter parts are introduced. One of the disadvantages, on the other hand, is that it is usually difficult to solve the old quantum canonical variables (q_r, p_s) in terms of the new ones (Q_i, P_j), thus prohibiting us to write the new Hamiltonian $K = H + \frac{\partial F^\dagger}{\partial t}$ in terms of (Q_i, P_j). But there are still quite large portions of quantum canonical transformations which are practically useful.

In this paper we consider quantum canonical transformations of the second kind with a special emphasis on the point transformations in relation to the generalized ladder operators. We show that non-Hermitian representations of the canonical commutation relations

$$[q_r, p_s] = i\delta_{rs}, \quad [q_r, q_s] = 0, \quad [p_r, p_s] = 0, \tag{1}$$

greatly simplify the quantization process of classical systems. It is also shown that the generalized ladder operators which are naturally associated with the non-Hermitian canonical variables allow us to solve eigenvalue problems quite elegantly.

General ideas on the quantum canonical transformations of the second kind with relation to coordinate reparametrizations are presented in the next section. The construction of the generalized ladder operators and a systematic way of solving eigenvalue equations are discussed in Sec. III. Some applications are shown in Sec. IV. The conclusion is given in Sec. V.

II. Quantum canonical transformations

In this paper we follow our previous notations which dealt on the general idea of the canonical transformations [4]. Let $|q'\rangle = |q'_1, \ldots, q'_f\rangle$ be a simultaneous eigenket of observables q_r, $r = 1, \ldots, f$, such that

$$q_r|q'\rangle = q'_r|q'\rangle, \tag{2}$$

$$\langle q'|q''\rangle = \frac{1}{\rho(q')}\delta(q' - q''), \tag{3}$$

$$\mathbf{1} = \int d^f q' \, |q'\rangle \rho(q')\langle q'|, \tag{4}$$

where we use the convention that various eigenvalues of an observable q_r are denoted by attaching primes such as q'_r, q''_r, etc. To investigate the general properties of coordinates transformations we introduce a set of functions $f_i(q'_1, \ldots, q'_f)$, $i = 1, \ldots, f$, such that

$$\det\left(\frac{\partial f_i}{\partial q'_r}\right) \neq 0. \tag{5}$$

Using $|q'\rangle$ and a generating function given by

$$F(q'_1, \ldots, q'_f, P'_1, \ldots, P'_f) = \sum_{i=1}^{f} f_i(q')P'_i, \tag{6}$$

we define another set of kets $|P'\rangle$ by

$$\langle q'|P'\rangle = e^{iF(q', P')}. \tag{7}$$

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For the reason of simplicity we consider only real functions f_i. From the completeness of $|q'\rangle$ it is easy to prove, after straight forward computations, that $|P'\rangle$ also forms a complete set and that the q-space and the P-space scale density functions $\rho(q')$ and $\rho(P')$ are

$$\rho(q') = \left| \det \left(\frac{\partial f_i}{\partial q'_r} \right) \right|, \quad \rho(P') = \frac{1}{2\pi}. \tag{8}$$

The completeness of $|P'\rangle$ allows us to define Hermitian operators P_i, $i = 1, \ldots, f$, such that

$$P_i |P'\rangle = P'_i |P'\rangle. \tag{9}$$

The physical meaning of P_i will become transparent when we interpret $q'_r \to P'_i$ as a part of a canonical transformation of the second kind corresponding to the classical generating function (6). The *well-ordered* generating operator which satisfies $\langle q'|F(q, P)|P'\rangle = \langle q'|F(q', P')|P'\rangle$ is

$$F(q, P) = \sum_{i=1}^{f} f_i(q) P_i. \tag{10}$$

Canonical operators p_r, and Q_i, which are defined by

$$\langle q'|p_r|P'\rangle = -i \frac{\partial}{\partial q'_r} \langle q'|P'\rangle, \tag{11}$$

$$\langle P'|Q_i|q'\rangle = i \frac{\partial}{\partial P'_i} \langle P'|q'\rangle, \tag{12}$$

have following forms

$$p_r = \frac{\partial F}{\partial q_r} = \sum_{i=1}^{f} \frac{\partial f_i}{\partial q_r} P_i, \tag{13}$$

$$Q_i = \frac{\partial F^\dagger}{\partial P_i} = f_i(q). \tag{14}$$

The equation (14) shows that the canonical transformation (7) corresponds to a reparametrization of the coordinates. It is important to notice that not only

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P_i, which from the very definition, but also Q_i, as it can be seen from (14), are Hermitian operators. But the Hermitian conjugation of p_r is

$$p_r^\dagger = \frac{1}{\rho(q)} p_r \rho(q).$$ (15)

It is true that instead of the non-Hermitian p_r one may choose a Hermitian combination $\frac{1}{2}(p_r + p_r^\dagger)$. But it is obvious that it is not imperative to use Hermitian representations even for the canonical variables which can not be measured with energy simultaneously. It will be soon clear that the freedom of choosing the non-Hermitian representations for some canonical variables makes us more versatile in various ways.

As an application of this idea consider a *Hermitian* Hamiltonian $H = \frac{1}{2}\sum_i P_i^2 + V(Q)$. For later conveniences we rewrite this as

$$H = \frac{1}{2} P_i^\dagger P_i + V(Q),$$ (16)

where we used the summation convention for the repeated indices. Inverting (13) we have

$$P_i = \frac{\partial q_r}{\partial Q_i} p_r.$$ (17)

The corresponding q-space Hamiltonian is

$$H = \frac{1}{2} p_r^\dagger \frac{\partial q_r}{\partial Q_i} \frac{\partial q_s}{\partial Q_i} p_s + V.$$ (18)

To simplify this equation we define a metric tensor $g_{ij}(Q) = \delta_{ij}$ in such a way that $ds^2 = g_{ij} dQ^i dQ^j$ is the invariant line element of coordinate transformations. The q-space metric tensor is, then,

$$g^{rs}(q) = \frac{\partial q_r}{\partial Q_i} \frac{\partial q_s}{\partial Q_i}.$$ (19)

It follows that the Hamiltonian operator (18) corresponding to the classical Hamiltonian $H = \frac{1}{2}p_r g^{rs} p_s + V$ simplifies to

$$H = \frac{1}{2}p_r^\dagger g^{rs} p_s + V. \tag{20}$$

Hamiltonian operators therefore can be unambiguously—that is, with no ordering ambiguity—constructed from the classical Hamiltonians as long as the non-Hermitian forms of canonical variables are used.

At this point we would like to emphasize that the q-space scale density function (8) obtainded from the completenesses of $|q'\rangle$ and $|P'\rangle$ is the same as that of the one obtained from (19)! It is reflected in the fact that the canonical transformation (7) is unitary and that (20) is Hermitian [4].

As a nontrivial illustration of this idea consider a free symmetrical top described by the following Hamiltonian function

$$H = \frac{p_\theta^2}{2I_1} + \frac{(p_\phi - p_\chi \cos\theta)^2}{2I_1 \sin^2\theta} + \frac{p_\chi^2}{2I_3}, \tag{21}$$

where θ, ϕ, and χ are the Euler angles describing the orientation of the symmetrical top, and I_1 and I_3 denote the moment of inertia along the principal axes. The ranges of the Euler anlges are

$$0 \leq \theta < \pi, \quad 0 \leq \phi < 2\pi, \quad 0 \leq \chi < 2\pi. \tag{22}$$

The metric tensor read off from (21) is

$$g^{rs} = \begin{pmatrix} \frac{1}{I_1} & 0 & 0 \\ 0 & \frac{1}{I_1 \sin^2\theta} & -\frac{\cos\theta}{I_1 \sin^2\theta} \\ 0 & -\frac{\cos\theta}{I_1 \sin^2\theta} & \frac{1}{I_3} + \frac{\cos^2\theta}{I_1 \sin^2\theta} \end{pmatrix}, \tag{23}$$

thus allowing us to write the scale density function $\rho(\theta\phi\chi)$ of the Euler angles,

$$\rho = \sqrt{\det g_{rs}} = \sqrt{I_1^2 I_3} \, \sin\theta. \tag{24}$$

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It is clear from (15) that p_ϕ and p_χ which are defined by

$$p_\phi = -i\frac{\partial}{\partial\phi}, \quad p_\chi = -i\frac{\partial}{\partial\chi}, \tag{25}$$

are Hermitian, but the Hermitian conjugation of p_θ is

$$p_\theta^\dagger = \frac{1}{\sin\theta}p_\theta\sin\theta. \tag{26}$$

The Hamiltonian operator is therefore

$$H = \frac{p_\theta^\dagger p_\theta}{2I_1} + \frac{(p_\phi - p_\chi\cos\theta)^2}{2I_1\sin^2\theta} + \frac{p_\chi^2}{2I_3}, \tag{27}$$

or, more explicitly,

$$H = -\frac{1}{2I_1}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} - \frac{1}{2I_1\sin^2\theta}\left(\frac{\partial}{\partial\phi} - \frac{\partial}{\partial\chi}\cos\theta\right)^2 - \frac{1}{2I_3}\frac{\partial^2}{\partial\chi^2} \tag{28}$$

which of course coincides with the known result [6].

III. Generalized ladder operators

The non-Hermitian realization of canonical variables and the general form (20) of the quantum mechanical Hamiltonian inspires us to introduce the following concept of the generalized ladder operators which is a generalization of the operator factorization concept of differential equations discussed by Infeld and Hull [7]. Suppose $\mathcal{E}$ is a Hibert space which can be decomposable into subspaces $\mathcal{E}(l)$,

$$\mathcal{E} = \bigoplus_l \mathcal{E}(l), \tag{29}$$

in the way that in the subspace $\mathcal{E}(l)$, which is usually an invariant subspace of a symmetry group $\mathcal{G}$ of the Hamiltonian, H is effectively $H(l)$. In addition to this suppose that $H(l)$ can be written as

$$H(l) = a(l)^\dagger a(l) + \nu(l) \tag{30}$$

$$= a(l+1)a(l+1)^\dagger + \nu(l+1) + \epsilon(l+1), \tag{31}$$

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where ν and ϵ are real numbers. Then one obtains following two equations

$$H(l-1)a(l) = a(l)\{H(l)+\epsilon(l)\}, \tag{32}$$

$$H(l+1)a(l+1)^\dagger = a(l+1)^\dagger\{H(l)-\epsilon(l+1)\}. \tag{33}$$

Whenever these hold solutions of the eingenvalue problems are automatic.

Suppose that $|El\tau\rangle$ which belonging to $\mathcal{E}(l)$ is an eigenstate of H with the eigenvalue E. τ is a set of any other quantum numbers which are irrelevant in this consideration. Multiplying this eigenket to the right of (32) one get

$$H(l-1)a(l)|El\tau\rangle = \{E+\epsilon(l)\}a(l)|El\tau\rangle, \tag{34}$$

showing that $a(l)|El\tau\rangle$ which belonging to $\mathcal{E}(l-1)$ is an eigenstate of H with the eigenvalue $E+\epsilon(l)$. This means that $a(l)$ is a descending ladder operator which raises energy by $\epsilon(l)$,

$$\cdots \overset{a(l-1)}{\longleftarrow} \mathcal{E}(l-1) \overset{a(l)}{\longleftarrow} \mathcal{E}(l) \overset{a(l+1)}{\longleftarrow} \mathcal{E}(l+1) \overset{a(l+2)}{\longleftarrow} \cdots \tag{35}$$

On the other hand, when one uses (33), one gets followng relation

$$H(l+1)a(l+1)^\dagger|El\tau\rangle = \{E-\epsilon(l+1)\}a(l+1)^\dagger|El\tau\rangle, \tag{36}$$

showing that $a(l+1)^\dagger|El\tau\rangle$ which belonging to $\mathcal{E}(l+1)$ is an eigenstate of H with the eigenvalue $E-\epsilon(l+1)$. This means that $a(l+1)^\dagger$ is an ascending ladder operator which lowers energy by $\epsilon(l+1)$,

$$\cdots \overset{a(l-1)^\dagger}{\longrightarrow} \mathcal{E}(l-1) \overset{a(l)^\dagger}{\longrightarrow} \mathcal{E}(l) \overset{a(l+1)^\dagger}{\longrightarrow} \mathcal{E}(l+1) \overset{a(l+2)^\dagger}{\longrightarrow} \cdots \tag{37}$$

With the help of these operators we have following two schems for solving eigenvalue equations.

Case-I. Strings of ascending states.

This is the case when there is a lower limit on the descension. In this case let $n = l_{min}$ and solve

$$a(n)|\text{lowest state}\rangle = 0. \tag{38}$$

Non-degeneracy of the solution gaurentees the non-degeneracy of the higher states. Normalizing this ket we recast this as $|nl_{min}\tau\rangle$ and define

$$E_{n,l_{min}} = \nu(n), \tag{39}$$

$$E_{n,l} = E_{n,l_{min}} - \sum_{k=l_{min}+1}^{l} \epsilon(k), \quad l > l_{min}. \tag{40}$$

As long as $E_{n,l-1} \neq \nu(l) + \epsilon(l)$, the vector

$$|nl\tau\rangle = \frac{a(l)^{\dagger}}{\sqrt{E_{n,l-1} - \nu(l) - \epsilon(l)}} |n, l-1, \tau\rangle \tag{41}$$

is a normalized eigenstate of H with the eigenvalue E_{nl}.

Case-II. Strings of descending states.

This is the case when there is an upper limit on the ascension. In this case let $n = l_{max}$ and solve

$$a(n+1)^{\dagger}|\text{highest state}\rangle = 0. \tag{42}$$

Properly normalizing this ket we rewrite this as $|nl_{max}\tau\rangle$ and define

$$E_{n,l_{max}} = \nu(n+1) + \epsilon(n+1), \tag{43}$$

$$E_{n,l} = E_{n,l_{max}} + \sum_{k=l+1}^{l_{max}} \epsilon(k), \quad l < l_{max}. \tag{44}$$

Then, as long as $E_{n,l+1} \neq \nu(l+1)$, the vector

$$|nl\tau\rangle = \frac{a(l+1)}{\sqrt{E_{n,l+1} - \nu(l+1)}} |n, l+1, \tau\rangle \tag{45}$$

is a normalized eingenstate of H with the eigenvalue $E_{n,l}$.

There is a third class of cases which have limitations both on the ascensions and descensions, but it is not necessary to consider them seperately. It can be considered as a special case of I or II.

IV. Applications of the generalized ladder operators

Consider a three dimensional spherically symmetric system described by the following Hamiltonian

$$H = \frac{p_r^\dagger p_r}{2} + \frac{\mathbf{L}^2}{2r^2} + V(r), \tag{46}$$

where $p_r^\dagger = \frac{1}{r^2} p_r r^2$ and $\mathbf{L}^2$ is the angular momentum operator

$$\mathbf{L}^2 = p_\theta^\dagger p_\theta + \frac{p_\phi^2}{\sin^2 \theta}. \tag{47}$$

Here the Hermitian conjugation of p_θ is $\frac{1}{\sin\theta} p_\theta \sin\theta$. Since this operator is invariant under the action of the $SO(3)$ transformations the Hilbert space $\mathcal{E}$ can be decomposed in terms of the eigenvector space $\mathcal{E}(l)$ of the angular momentum operator,

$$\mathcal{E} = \bigoplus_{l=0}^{\infty} \mathcal{E}(l). \tag{48}$$

(Using our generalized ladder operator technique one may solve the eigenvalue problem for $\mathbf{L}^2$ given by (47). This interesting digression is presented in the Appendix.) The Hamiltonian H in $\mathcal{E}(l)$ is

$$H(l) = \frac{p_r^\dagger p_r}{2} + \frac{l(l+1)}{2r^2} + V(r). \tag{49}$$

To be more specific, consider a three dimensional Harmonic oscillator described by the potential $V(r) = \frac{1}{2}\omega^2 r^2$. In this case we have

$$a(l) = \frac{1}{\sqrt{2}} \left(ip_r + \frac{l+1}{r} - \omega r \right), \tag{50}$$

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$$\nu(l) = \left(l - \frac{1}{2}\right)\omega, \tag{51}$$

$$\epsilon(l) = \omega. \tag{52}$$

To get finite-norm eigenvectors one should assume the Case-II. Putting $n = l_{max}$ we solve (42) obtaining the following normalized eigenstate

$$\psi_{n,l_{max},m}(r\theta\phi) = \sqrt{\frac{\omega^{n+\frac{3}{2}}n!}{\sqrt{\pi}(2n+1)!}}\, 2^{n+1}\, r^n e^{-\frac{\omega}{2}r^2}\, Y_{l_{max},m}(\theta\phi). \tag{53}$$

The eigenvalue of this state is $E_{n,l_{max}} = (n + \frac{3}{2})\omega$. The normalized eigenstate for $0 \leq l < n$ is

$$\psi_{nlm} = \sqrt{\frac{\omega^{l+\frac{3}{2}}n!}{\sqrt{\pi}(n-l)!(2n+1)!}}\, 2^{l+1}\, e^{\frac{\omega}{2}r^2}\frac{1}{r^{l+1}}\left(\frac{1}{r}\frac{d}{dr}\right)^{n-l}\left(r^{2n+1}e^{-\omega r^2}\right) Y_{lm}, \tag{54}$$

and the corresponging eigenvalue is

$$E_{nl} = (2n - l + \frac{3}{2})\omega. \tag{55}$$

This way of solving the eigenvalue problem and the resulting forms of the wave functions are much nicer than the one from the series expansions and the Laguerre functions.

One may apply this idea to the eigenvalue problem of the hydrogen atom. But it is rather trivial, and we turn to the spinning top problem whose corresponding Hamiltonian is given by (27). Since the eigenvalues of the Hermitian operators given by (25) are $p_\phi = m$ and $p_\chi = l$ with integer m and l, the Hilbert space $\mathcal{E}$ can be decomposed as

$$\mathcal{E} = \bigoplus_{l,m=-\infty}^{\infty} \mathcal{E}(l,m). \tag{56}$$

The Hamiltonian in this subspace is

$$H(l,m) = \frac{p_\theta^\dagger p_\theta}{2I_1} + \frac{(m - l\cos\theta)^2}{2I_1\sin^2\theta} + \frac{l^2}{2I_3}. \tag{57}$$

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Then defining

$$a(l, m) = \frac{1}{\sqrt{2I_1}} \left(ip_\theta + \frac{l\cos\theta - m}{\sin\theta} \right), \tag{58}$$

$$\nu(l) = \frac{1}{2} \left(\frac{l^2}{I_3} - \frac{l}{I_1} \right), \tag{59}$$

$$\epsilon(l) = -\frac{1}{2} \left(\frac{1}{I_3} - \frac{1}{I_1} \right) (2l - 1), \tag{60}$$

one can prove that the Hamiltonian operator $H(l, m)$ which acts in the subspace $\mathcal{E}(l, m)$ can be written as

$$H(l, m) = a(l, m)^\dagger a(l, m) + \nu(l) \tag{61}$$

$$= a(l+1, m) a(l+1, m)^\dagger + \nu(l+1) + \epsilon(l+1). \tag{62}$$

These show that the raising and lowering operations do not change the quantum number m. The normalized wave function $\psi_{n, l_{max}, m}(\theta\phi\chi)$ for $l = l_{max} \equiv n$, which can be solved from (42), is

$$\psi_{n, l_{max}, m} = \frac{1}{4\pi} \sqrt{\frac{2(2n+1)!}{(n+m)!(n-m)!}} \left(\cos\frac{\theta}{2} \right)^{n+m} \left(\sin\frac{\theta}{2} \right)^{n-m} e^{in\chi + im\phi}. \tag{63}$$

By checking the termination point of the descension and the behavior of the wave function at $\theta = 0$ and π, it is not difficult to show that

$$-l_{min} = l_{max} \geq |m|. \tag{64}$$

The energy eigenvalue for general l and m is

$$E_{nl} = \frac{n(n+1)}{2I_1} + \left(\frac{1}{I_3} - \frac{1}{I_1} \right) \frac{l^2}{2}, \tag{65}$$

which is in good agreement with known result [6, 8]. The wave function for this energy level can be obtained from (45).

V. Conclusion

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Non-Hermitian representations of the canonical commutation relations, rather than Hermitian ones, are much more useful for both quantizing classical systems and solving eigenvalue equations. To make it clear we introduced quantum canonical transformations of the second type with a special interest on those relations to the generalized ladder operators. The usual operator ordering problems, which are inevitable when there are no indisputable principles, do not occure when one follows the mixed matrix element technique of the canonical transformations. This means that for a given classical Hamiltonian one may write the Hamiltonian operator immediatly. Another advantage of the non-Hermitian realization is that it is possible to introduce the generalized ladder operators naturally which may greatly simplify for solving eigenvalue equations. It is certain that one can not solve *all* the eigenvalue equations in this way. But it is quite probable that whenever there is a dynamical symmetry associated with a Hamiltonian, the relevant generalized ladder opeartors can be found.

Applications to quantum field theories are open when one uses the functional Schrödinger equation formulations for these. Our further investigation is aimed on this project.

Acknowledgments

APPENDIX: The eigenvalue problems of the angular momentum operator using the generalized ladder operators

Consider the following angular momentum operator

$$\mathbf{L}^2 = p_\theta^\dagger p_\theta + \csc^2 \theta \, p_\phi^2, \tag{A1}$$

where $0 \leq \theta < \pi$ and $0 \leq \phi < 2\pi$ and

$$p_\theta^\dagger = \frac{1}{\sin \theta} \, p_\theta \sin \theta. \tag{A2}$$

The eigenvalues of the Hermitian p_ϕ are integers which are denoted generally as m. We decompose the Hilbert space $\mathcal{E}$ of $\mathbf{L}^2$ in terms of the eigenvector spaces $\mathcal{E}(m)$ of p_ϕ,

$$\mathcal{E} = \bigoplus_{m=-\infty}^{\infty} \mathcal{E}(m), \tag{A3}$$

in such a way that in each subspace $\mathbf{L}^2$ becomes

$$\mathbf{L}^2(m) = p_\theta^\dagger p_\theta + m^2 \cot^2 \theta + m^2. \tag{A4}$$

Here we used the fact $\csc^2 \theta = \cot^2 \theta + 1$. It is easy to show that when one defines $a(m)$ as

$$a(m) = i p_\theta + m \cot \theta, \tag{A5}$$

it is in fact a generalized descending ladder opeartor. The associated relevant quantities are

$$\nu(m) = m(m-1), \tag{A6}$$

$$\epsilon(m) = 0. \tag{A7}$$

The normalized eigenstate for $l = m_{max}$, which can be solved from (42), is

$$\psi_{l,m_{max}}(\theta, \phi) = (-)^l \frac{1}{2^l l!} \sqrt{\frac{(2l+1)!}{4\pi}} e^{il\phi} \sin^l \theta, \quad l = 0, 1, 2, \ldots \qquad (A8)$$

By the action of $a(l)$ on this state the eigenvalue of $\mathbf{L}^2$ does not change. But for p_ϕ it changes from l to $l-1$. This process of descension terminates at $m_{min} = -l$. That is, our string of descending states exactly coincides with Y_{lm}, the spherical harmonics.

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